WEAKLY COMPACT SETS IN BANACH SPACES

A dissertation submitted to the
Kent State University Graduate College
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

by

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August 1992

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Director of Dissertation: Dr. Joseph Diestel

The classical theorem of Dunford and Pettis , identifies the bounded, uniformly integrable subsets of $L^1(\mu)$ with the relatively weakly compact sets. Another characterization of uniform integrability is given in a theorem of De La Vallée Poussin which states that *a subset $\mathcal{K}$ of $L^1(\mu)$ is bounded and uniformly integrable if and only if it is a bounded subset of some Orlicz space L_F^*.* We refine and improve this theorem in several directions. The theorem of De La Vallée Poussin does not, for instance, specify just how well the function F can be chosen. It gives little additional information in case the set in question is relatively norm compact in $L^1(\mu)$. Finally it gives no information on the structure of the set in the corresponding Banach space of F-integrable functions. More specifically we establish the fact that *a subset $\mathcal{K}$ of L^1 is relatively compact if and only if there is an N-function $F \in \Delta'$ so that $\mathcal{K}$ is relatively compact in L_F^** . Furthermore we prove that *a subset $\mathcal{K}$ of L^1 is relatively weakly compact if and only if there is an N-function $F \in \Delta'$ so that $\mathcal{K}$ is relatively weakly compact in L_F^** . We then go on to show that a large class of non-reflexive Orlicz spaces has the weak Banach-Saks property, by establishing a result for these spaces, very similar to the Dunford-Pettis theorem for L^1. Finally we investigate some similarities of these spaces, with the space $L^1(\mu)$. Kadec and Pelczýnski have shown that every non-reflexive subspace of $L^1(\mu)$ contains a copy of l_1 complemented in $L^1(\mu)$. On the other hand Rosenthal investigated the structure of reflexive subspaces of $L^1(\mu)$ and proved that such subspaces, have non-trivial type. We show the same facts to hold true, for the special class of non-reflexive Orlicz spaces, we have been investigating.

AKNOWLEDGEMENTS

At this point I want to thank all the people that helped and encouraged me during the past years. I would like to express my most sincere gratitude to my advisor Joe Diestel. His enthusiasm, optimism, advice and support made an enormous difference in my life as a mathematician and as a person in general.

I would also like to thank Richard Aron and Andrew Tonge for talking to me about a variety of problems as well as for conducting a multitude of stimulating courses and seminars, from which I benefited greatly.

I take now this opportunity to also express my deepest appreciation to my colleague Paul Abraham who patiently listened to many arguments, offered many suggestions and taught me a great deal through seminars and private conversations.

Thanks also to Miguel Lacruz, Chris Lennard, Tony Weston, Carmen Romero and Guillermo Curbera (Bill from Seville) for the many interesting discussions that made the environment in Kent so positive for mathematical development. My undergraduate teachers, Nicholas Bezak, Ben Freed, Stephen Gendler, Mike Osessia and especially Roger Engle for transforming my world of mathematics, from a necessary nuisance, to an everlasting passion.

I cannot find the right words to thank my father, mother and brother for all they have done for me. So I will just say thanks and I hope that they understand. Special thanks also to Francie (Fritz) Gifford for her kindness compassion, understanding and encouragement during some very difficult moments of my life.

Finally the support of the Department of Mathematics and Computer Science and Graduate College at Kent State University as well as that of Motor Oil Hellas is gratefully acknowledged.

To Jane Elizabeth

Chapter 0

INTRODUCTION

The notation used throughout this dissertation is fairly standard. A close model is the notation in Diestel [6].

(Ω, Σ, μ) will denote a non-atomic probability space and $L^p(\mu)$ will denote the Banach space of (equivalence classes of) measurable, real-valued functions on Ω, whose p-th power is μ-integrable. $\mathbb{R}$ denotes the set of real numbers. The symbol $\|\cdot\|$ is used to denote a Banach space norm. Sometimes subscripts are placed in the norm symbol, in order to identify the space on which the norm is taken. The symbol $\chi_{(\cdot)}$ is used to denote characteristic functions of sets. That is for a set A in Σ, χ_A is a real valued function defined on Ω by

$$\chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

Finally given a measurable function f on Ω and a real number a, the probabilistic notation $[f > a]$, $[f \geq a]$, $[f < a]$, $[f \leq a]$ and $[f = a]$ is used to describe the sets of all elements $\omega \in \Omega$ for which $f(\omega) > a$, $f(\omega) \geq a$, $f(\omega) < a$, $f(\omega) \leq a$ and $f(\omega) = a$ respectively.

For the various concepts in Banach space theory and measure theory, not defined explicitly, the reader should consult Banach [4], Dunford and Schwartz [9], Diestel [6], Diestel and Uhl [7], Rudin [28], [29] and Halmos [11].

Recall that a subset $\mathcal{K}$ of $L^1(\mu)$ is called uniformly integrable if given $\varepsilon > 0$ there is a $\delta > 0$ so that $\sup\{\int_E |f| d\mu : f \in \mathcal{K}\} < \varepsilon$ whenever $\mu(E) < \delta$. Alternatively $\mathcal{K}$ is bounded and uniformly integrable if and only if given $\varepsilon > 0$ there is an $N > 0$ so that

$$\sup\left\{\int_{[\,|f|>c\,]} |\, f \,|\, d\mu : f \in \mathcal{K}\right\} < \varepsilon \text{ whenever } c \geq N.$$

The classical theorem of Dunford and Pettis [6, page 93], identifies the bounded, uniformly integrable subsets of $L^1(\mu)$ with the relatively weakly compact sets. Another characterization of uniform integrability is given in a theorem of De La Vallée Poussin [22, pages 19-20], which states that *a subset $\mathcal{K}$ of $L^1(\mu)$ is bounded and uniformly integrable if and only if there is an N-function F so that* $\sup\{\int F(f)d\mu : f \in \mathcal{K}\} < \infty$. A function $F : \mathbb{R} \to [0,\infty)$ is called an *N-function*, if it is continuous, even and convex with $\lim_{t\to\infty} \frac{F(t)}{t} = \infty$ and $\lim_{t\to 0} \frac{F(t)}{t} = 0$. Given an N-function F, the function G defined by $G(x) = \sup\{t|x| - F(t) : t \geq 0\}$ is an N-function, called the complement of F.

De La Vallée Poussin's theorem is the focal point of Chapter 1 and the main reason that the other chapters exist. We refine and improve this theorem in several directions. The theorem of De La Vallée Poussin does not, for instance, specify just how well the function F can be chosen. It gives little additional information in case the set in question is relatively norm compact in $L^1(\mu)$. Finally it gives no information on the structure of the set in the corresponding Banach space of F-integrable functions. Such a space is called an *Orlicz space.* Given an N-function F, the Orlicz space determined by F is defined by

$$L_F^* = \{f \ \textit{measurable} \ : \exists\, c > 0 \ \textit{such that} \ \int_\Omega F(cf(\omega))d\mu(\omega) < \infty\},$$

where the usual identification of functions differing only on a set of measure zero, takes place. The norm of an element $f \in L_F^*$ is given by

$$\|f\|_F = \inf\{\frac{1}{c}(1 + \int F(cf)d\mu) \ : \ c > 0\}.$$

It is worth mentioning at this stage that if $1 < p < \infty$ and F is defined by $F(t) = |t|^p$, then L_F^* is just the familiar L^p space. Most of the results in this dissertation deal with Orlicz spaces whose generating N-functions satisfy the Δ_2 or Δ' conditions. We say that an N-function F satisfies the Δ_2 condition ($F \in \Delta_2$) if there is a constant K so that $F(2x) \leq KF(x)$ for large values of x. An N-function F satisfies the Δ' condition ($F \in \Delta'$) if there is a constant K so that $F(xy) \leq KF(x)F(y)$ for large values of x and y.

More specifically in Section 1.4 we establish the fact that *a subset $\mathcal{K}$ of L^1 is relatively compact if and only if there is an N-function $F \in \Delta'$ so that $\mathcal{K}$ is relatively compact in L_F^* (Theorem 1.4.3)*. Furthermore in the same Section we prove that *a subset $\mathcal{K}$ of L^1 is relatively weakly compact if and only if there is an N-function $F \in \Delta'$ so that $\mathcal{K}$ is relatively weakly compact in L_F^* (Theorem 1.4.7)*. In establishing this last result, a weak compactness criterion for Orlicz spaces was used (Theorem 1.4.5). The technique employed to prove this criterion was mainly averaging. Thus the natural question of Orlicz spaces and their relationship to Banach-Saks types of properties arises.

Recall that a Banach space X has the Banach-Saks (weak Banach-Saks) property if every bounded (weakly null) sequence in X has a subsequence, each subsequence of which, has norm convergent arithmetic means. Banach and Saks have shown in [5] that L^p, for $p > 1$, has the Banach-Saks property, while Szlenk in [31] established the fact that L^1 has the weak Banach-Saks property. Nishiura and Waterman showed in [24] that Banach spaces with the Banach-Saks property are reflexive. On the other hand Kakutani in [15] proved that uniformly convexifiable spaces have the Banach-Saks property. Baernstein in [3] gave the first example of a reflexive Banach space that fails the Banach Saks property. Furthermore Schreier in [30] established the fact that $C[0, 1]$, fails the weak Banach-Saks property. Akimovich in [1] has shown that reflexive Orlicz spaces are uniformly convexifiable and so they have the Banach-Saks property.

In Chapter 2 we show that a large class of non-reflexive Orlicz spaces[1] has the weak Banach-Saks property, by establishing a result for these spaces, very similar to the Dunford-Pettis Theorem for L^1. Before we mention the results in Chapter 2, we need to recall, that

[1] The idea for studying this class comes from [18]

a subset $\mathcal{K}$ of an Orlicz space L_F^*, has equi-absolutely continuous norms, if given $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\sup\{\|\chi_A \cdot f\|_F \;:\; f \in \mathcal{K}\} < \varepsilon$$

for all measurable sets A with $\mu(A) < \delta$.

In Section 2.1 we show that *if $F \in \Delta_2$ and its complement G satisfies $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ for some $c > 0$, then any weakly null sequence in L_F^* has equi-absolutely continuous norms (Theorem 2.1.3).* As a corollary to this theorem we have that *if F is as above then a bounded set in L_F^* is relatively weakly compact if and only if it has equi-absolutely continuous norms (Corollary 2.1.4).* Furthermore, under the same hypothesis *L_F^* has the weak Banach-Saks property (Corollary 2.1.5).* These results complement the ones of T. Ando in [2]. In Section 2.2 we give an application in convex function theory. Specifically we answer negatively the following question posed in [17, page 30]: Given an N-function $F \in \Delta'$, is it possible to find an N-function H equivalent to F so that H satisfies the Δ' condition, for all real x, y ?

Having this 'Dunford-Pettis' type of result for this special class of non-reflexive Orlicz spaces, we continue on to Chapter 3, where we investigate some similarities of these spaces, with the space $L^1(\mu)$. Kadec and Pelczýnski in [13] have shown that every non-reflexive subspace of $L^1(\mu)$ contains a copy of l_1 complemented in $L^1(\mu)$. On the other hand Rosenthal in [27] investigated the structure of reflexive subspaces of $L^1(\mu)$ and proved that such subspaces, have non-trivial type. Recall that a Banach space X has type p for some $1 < p \leq 2$, if there is a $K > 0$, so that

$$\left(\int_0^1 \|\sum_{i=1}^n r_i(t)x_i\|^p dt\right)^{\frac{1}{p}} \leq K \cdot \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}},$$

where (r_i) denotes the sequence of Rademacher functions[2] and $x_1, \ldots, x_n$ are arbitrary

[2]For a positive integer n, $r_n : [0,1] \to \{-1,1\}$ is defined by

$$r_n(t) = \begin{cases} -1 & \text{if } t = 1 \\ (-1)^{i-1} & \text{if } t \in [\frac{i-1}{2^n}, \frac{i}{2^n}), \text{ where } i = 1, \ldots, 2^n \end{cases}$$

elements of the Banach space X.

In Chapter 3 we show the same facts to hold true for the special class of non-reflexive Orlicz spaces we have been investigating. In particular, in Section 3.1 we show that *if F is an N-function in Δ_2 with its complement G satisfying* $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ *then every non-reflexive subspace of L_F^*, contains a copy of l_1 complemented in L_F^* (Theorem 3.1.4).* Furthermore in Section 3.3 we show that *if F is an N-function in Δ_2 with its complement G satisfying* $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ *then every reflexive subspace of L_F^* has non trivial type (Theorem 3.3.3).*

Chapter 1

DE LA VALLÉE POUSSIN'S THEOREM REVISITED

1.1 Uniform integrability and De La Vallée Poussin's Theorem.

Definition 1.1.1 *A subset $\mathcal{K}$ of $L^1(\mu)$ is called uniformly integrable if*

$$\lim_{c\to\infty} \sup\{\int_{[\,|f|\geq c\,]} |\, f\,|\; d\mu : f \in \mathcal{K}\} = 0\ .$$

That is given $\varepsilon > 0$ there is a $c_\varepsilon > 0$ so that for each $f \in \mathcal{K}$ and each $c \geq c_\varepsilon$ we have

$$\int_{[\,|f|\geq c\,]} |\, f\,|\; d\mu < \varepsilon\ .$$

Another way of defining uniform integrability is described in the following proposition:

Proposition 1.1.1 *A subset $\mathcal{K}$ of $L^1(\mu)$ is uniformly integrable if and only if it is L^1 - bounded and for each $\varepsilon > 0$ there is a $\delta > 0$ so that $\sup\{\int_A |\, f\,|\; d\mu : f \in \mathcal{K}\} < \varepsilon$ for all $A \in \Sigma$ with $\mu(A) < \delta$.*

Proof : First note that for all measurable A, $f \in \mathcal{K}$, $c > 0$ we have

$$\int_A |\, f\,|\; d\mu = \int_{A\cap[\,|f|<c\,]} |\, f\,|\; d\mu + \int_{A\cap[\,|f|\geq c\,]} |\, f\,|\; d\mu \leq c\mu(A) + \int_{[\,|f|\geq c\,]} |\, f\,|\; d\mu\ .$$

Fix $\varepsilon > 0$ and choose $c_0 > 0$ so that $\sup\{\int_{[\,|f|\geq c\,]} |\, f\,|\; d\mu : f \in \mathcal{K}\} < \frac{\varepsilon}{2}$ whenever $c \geq c_0$. Then for all $f \in \mathcal{K}$ we have

$$\int_\Omega |\, f\,|\; d\mu \leq c_0\mu(\Omega) + \int_{[\,|f|\geq c_0\,]} |\, f\,|\; d\mu \leq c_0 + \frac{\varepsilon}{2}$$

and thus $\mathcal{K}$ is L^1 bounded. Now let $0 < \delta < \frac{\varepsilon}{2c_0}$. Then for all measurable A with $\mu(A) < \delta$ and all $f \in \mathcal{K}$ we have

$$\int_A |\, f\,|\; d\mu \leq c_0\mu(A) + \int_{[\,|f|\geq c_0\,]} |\, f\,|\; d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ .$$

We now prove the converse. Fix $\varepsilon > 0$ and choose $\delta > 0$ so that $\sup\{\int_A \mid f \mid d\mu : f \in \mathcal{K}\} < \varepsilon$ whenever A is measurable with $\mu(A) < \delta$. Let $M = \sup\{\int_\Omega \mid f \mid d\mu : f \in \mathcal{K}\}$ and choose $c_0 > 0$ so that $\frac{M}{c_0} < \delta$. Then for all $f \in \mathcal{K}$ and all $c \geq c_0$ we have

$$\mu([\mid f \mid \geq c\,]) \leq \frac{1}{c} \int_{[\,|f| \geq c\,]} \mid f \mid d\mu \leq \frac{M}{c_0} < \delta\,.$$

So $\int_{[\,|f| \geq c\,]} \mid f \mid d\mu < \varepsilon$ and so we are done. ∎

The following well known theorem of Dunford and Pettis, gives some more insight to the notion of uniform integrability.

Theorem 1.1.2 (Dunford-Pettis) *: A subset $\mathcal{K}$ of $L^1(\mu)$ is uniformly integrable if and only if it is relatively weakly compact.*

A proof of this theorem can be found in [6, page 93].

Yet another characterization of uniformly integrable sets is an old theorem that finds its roots in Harmonic Analysis and Potential theory. It is due to De La Vallée Poussin. Since it is this theorem that we deal with in this chapter, we state and prove this result in detail (see [22, pages 19–20]).

Theorem 1.1.3 (De La Vallée Poussin) *A subset $\mathcal{K}$ of $L^1(\mu)$ is uniformly integrable if and only if there is a non-negative and convex function Q with $\lim_{t\to\infty} \frac{Q(t)}{t} = \infty$ so that*

$$\sup\{\int_\Omega Q(\mid f \mid) d\mu : f \in \mathcal{K}\} < \infty\,.$$

Proof : Suppose that $\mathcal{K}$ is a uniformly integrable subset of $L^1(\mu)$. We will construct a non-negative and non-decreasing function q that is constant on $[n, n+1)$ for $n = 0, 1, \ldots$ with $\lim_{t\to\infty} q(t) = \infty$ and we will set $Q(x) = \int_0^x q(t)dt$ for $x > 0$. Use the hypothesis to choose a subsequence (c_n) of the positive integers so that

$$\sup\{\int_{[\,|f| \geq c_n\,]} \mid f \mid d\mu : f \in \mathcal{K}\} < \frac{1}{2^n} \;\; \forall\, n = 1, 2, \ldots\,.$$

Then for each $f \in \mathcal{K}$ and all $n = 1, 2, \ldots$ we have

$$\begin{aligned}
\int_{[\,|f| \geq c_n\,]} |\, f \,|\, d\mu &= \sum_{m=c_n}^{\infty} \int_{[\,m \leq |f| < m+1\,]} |\, f \,|\, d\mu \\
&\geq \sum_{m=c_n}^{\infty} m\mu([\,m \leq |\, f \,| < m+1\,]) \\
&\geq \sum_{m=c_n}^{\infty} \mu([\,|\, f \,| \geq m\,]) \ .
\end{aligned}$$

So for all $f \in \mathcal{K}$ we have

$$\sum_{n=1}^{\infty} \sum_{m=c_n}^{\infty} \mu([\,|\, f \,| \geq m\,]) \leq 1 \ .$$

Now for $m = 1, 2, \ldots$ let q_m be the number of the positive integers n, for which $c_n \leq m$. Then $q_m \nearrow \infty$. Furthermore observe that

$$\sum_{n=1}^{\infty} \sum_{m=c_n}^{\infty} \mu([\,|\, f \,| \geq m\,]) = \sum_{k=1}^{\infty} q_k \mu([\,|\, f \,| \geq k\,]) \ .$$

Let $q_0 = 0$ and define $q(t) = q_n$ if $t \in [n, n+1)$ for $n = 0, 1, 2, \ldots$. Then if $Q(x) = \int_0^x q(t)dt$ we have

$$\begin{aligned}
\int_{\Omega} Q(|\, f \,|) d\mu &= \sum_{n=0}^{\infty} \int_{[\,n \leq |f| < n+1\,]} Q(|\, f \,|) d\mu \\
&\leq \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{n} q_m\Big) \cdot \mu([\,n \leq |\, f \,| < n+1\,]) \\
&= q_0 \cdot \mu([\,0 \leq |\, f \,| < 1\,]) + (q_0 + q_1) \cdot \mu([\,1 \leq |\, f \,| < 2\,]) + \cdots \\
&= \sum_{n=0}^{\infty} q_n \mu([\,|\, f \,| \geq n\,]) \\
&\leq 1 \ .
\end{aligned}$$

So $\sup\{\int_{\Omega} Q(|\, f \,|) d\mu : f \in \mathcal{K}\} < \infty$.

To see that Q is convex, fix $0 \leq x_1 < x_2$. We then have

$$\begin{aligned}
Q(\frac{1}{2}(x_1 + x_2)) &= \int_0^{\frac{1}{2}(x_1+x_2)} q(t)dt \\
&= \int_0^{x_1} q(t)dt + \int_{x_1}^{\frac{1}{2}(x_1+x_2)} q(t)dt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{x_1} q(t)dt + \frac{1}{2}\int_{x_1}^{\frac{1}{2}(x_1+x_2)} q(t)dt + \frac{1}{2}\int_{\frac{1}{2}(x_1+x_2)}^{x_2} q(t)dt \\
&= \frac{1}{2}\int_0^{x_1} q(t)dt + \frac{1}{2}\int_0^{x_2} q(t)dt \\
&= \frac{1}{2}(Q(x_1)+Q(x_2)) \; .
\end{aligned}$$

Finally observe that

$$Q(x) = \int_0^x q(t)dt \geq \int_{\frac{x}{2}}^x q(t)dt \geq \frac{x}{2}q(\frac{x}{2})$$

and thus $\frac{Q(x)}{x} \geq \frac{1}{2}q(\frac{x}{2}) \to \infty$ as $x \to \infty$.

We now prove the converse. Let $M = \sup\{\int_\Omega Q(\mid f \mid)d\mu : f \in \mathcal{K}\}$. Let $\varepsilon > 0$ and choose $c_0 > 0$ so that $\frac{Q(t)}{t} > \frac{M}{\varepsilon}$ whenever $t \geq c_0$. Then for $f \in \mathcal{K}$ and $c \geq c_0$ we have that $\mid f \mid < \frac{\varepsilon}{M}Q(\mid f \mid)$ on the set $[\,\mid f \mid \geq c\,]$. Thus

$$\int_{[\,|f|\geq c\,]} \mid f \mid d\mu \leq \frac{\varepsilon}{M}\int_{[\,|f|\geq c\,]} Q(\mid f \mid)d\mu \leq \frac{\varepsilon}{M}M = \varepsilon$$

and so we are done. ∎

1.2 Some facts about N-Functions

In this section we will summarize the necessary facts about a special class of convex functions called N-functions. For a detailed account of these facts, the reader could consult the first chapter in [17].

Definition 1.2.1 *Let $p : [0,\infty) \to [0,\infty)$ be a right continuous, monotone increasing function with*

1. $p(0) = 0$;
2. $\lim_{t\to\infty} p(t) = \infty$;
3. $p(t) > 0$ *whenever* $t > 0$;

then the function defined by

$$F(x) = \int_0^{|x|} p(t)dt$$

is called an N-function.

The following proposition gives an alternative view of N-functions.

Proposition 1.2.1 *The function F is an N-function if and only if F is continuous, even and convex with*

1. $\lim_{x\to 0} \frac{F(x)}{x} = 0$;

2. $\lim_{x\to\infty} \frac{F(x)}{x} = \infty$;

3. $F(x) > 0$ *if* $x > 0$.

Definition 1.2.2 *For an N-function F define*

$$G(x) = \sup\{t|x| - F(t) : t \geq 0\} .$$

Then G is an N-function and it is called the complement of F.

Observe that F is the complement of its complement G.

Theorem 1.2.2 (Young's Inequality) *If F and G are two mutually complementary N-functions then*

$$xy \leq F(x) + G(y) \quad \forall x, y \in \mathbb{R} .$$

Proposition 1.2.3 *The composition of two N-functions is an N-function. Conversely every N-function can be written as a composition of two other N-functions.*

The following material deals with the comparative growth of N-functions.

Definition 1.2.3 *For N-functions F_1, F_2 we write $F_1 \prec F_2$ if there is a $K > 0$ so that $F_1(x) \leq F_2(Kx)$ for large values of x. If $F_1 \prec F_2$ and $F_2 \prec F_1$ then we say that F_1 and F_2 are equivalent.*

Proposition 1.2.4 *If $F_1 \prec F_2$ then $G_2 \prec G_1$, where G_i is the complement of F_i. In particular if $F_1(x) \leq F_2(x)$ for large values of x then $G_2(x) \leq G_1(x)$ for large values of x.*

Definition 1.2.4 *A convex function Q is called the principal part of an N-function F, if $F(x) = Q(x)$ for large x.*

Proposition 1.2.5 *If Q is convex with $\lim\limits_{x\to\infty} \dfrac{Q(x)}{x} = \infty$ then Q is the principal part of some N-function.*

Definition 1.2.5 *An N-function F is said to satisfy the Δ_2 condition $(F \in \Delta_2)$ if $\limsup_{x\to\infty} \frac{F(2x)}{F(x)} < \infty$. That is, there is a $K > 0$ so that $F(2x) \leq KF(x)$ for large values of x.*

Definition 1.2.6 *An N-Function F is said to satisfy the Δ' condition $(F \in \Delta')$ if there is a $K > 0$ so that $F(xy) \leq KF(x)F(y)$ for large values of x and y.*

Definition 1.2.7 *An N-function F is said to satisfy the Δ_3 condition $(F \in \Delta_3)$ if there is a $K > 0$ so that $xF(x) \leq F(Kx)$ for large values of x.*

Definition 1.2.8 *An N-function F is said to satisfy the Δ^2 condition $(F \in \Delta^2)$ if there is a $K > 0$ so that $(F(x))^2 \leq F(Kx)$ for large values of x.*

Theorem 1.2.6 *Let F be an N-function and let G be its complement; then the following hold.*

- *If $F \in \Delta'$ then $F \in \Delta_2$.*
- *If $F \in \Delta_3$ then its complement $G \in \Delta_2$.*
- *If $F \in \Delta^2$ then its complement $G \in \Delta'$.*
- *If $F \in \Delta_2$ then there is a $p > 1$ so that if $H(x) = |x|^p$ then $F \prec H$.*

Finally the classes Δ', Δ_2, Δ_3 and Δ^2 are preserved under equivalence of N-functions.

1.3 Some facts about Orlicz Spaces

In this section we summarize the necessary definitions and results about Orlicz spaces. A detailed account can be found in chapter two of [17].

Definition 1.3.1 *For an N-function F and a measurable f define*

$$\mathbf{F}(f) = \int F(f) d\mu.$$

Let $L_F = \{f \text{ measurable} : \mathbf{F}(f) < \infty\}$. *If G denotes the complement of F let*

$$L_F^* = \{f \text{ measurable} : |\int fg d\mu| < \infty \quad \forall g \in L_G\} \ .$$

The collection L_F^* is then a linear space. For $f \in L_F^*$ define

$$\|f\|_F = \sup\{|\int fg d\mu| : \mathbf{G}(g) \leq 1\} \ .$$

Then $(L_F^*, \|\cdot\|_F)$ is a Banach space, called an Orlicz space.

The following theorem establishes the fact that an Orlicz space is a dual space.

Theorem 1.3.1 *Let F be an N-function and let E_F be the closure of the bounded functions in L_F^*. Then the conjugate space of E_F is L_G^*, where G is the complement of F.*

Theorem 1.3.2 *Let F be an N-function and G be its complement. Then the following statements are equivalent:*

1. $L_F^* = E_F$.

2. $L_F^* = L_F$.

3. *The dual of* L_F^* *is* L_G^*.

4. $F \in \Delta_2$.

Theorem 1.3.3 (Hölder's Inequality) *For $f \in L_F^*$ and $g \in L_G^*$ we have*

$$\int |fg| d\mu \leq \|f\|_F \cdot \|g\|_G \,.$$

Theorem 1.3.4 *If $f \in L_F^*$ then*

$$\|f\|_F = \inf\left\{\frac{1}{k}(1 + \mathbf{F}(kf)) : k > 0\right\}.$$

It follows then that $f \in L_F^*$ if and only if there is $c > 0$ so that $\mathbf{F}(cf) < \infty$.

Proposition 1.3.5 *If $\|f\|_F \leq 1$ then $f \in L_F$ and $\mathbf{F}(f) \leq \|f\|_F$.*

Comparison of N-functions, gives rise to the following result concerning their corresponding Orlicz spaces.

Proposition 1.3.6 *If $F_1 \prec F_2$ then $L_{F_2}^* \subset L_{F_1}^*$ and the inclusion mapping is continuous.*

Definition 1.3.2 *We say that a collection $\mathcal{K} \subset L_F^*$ has equi-absolutely continuous norms if*

$$\forall\, \varepsilon > 0 \,\exists\, \delta > 0 \text{ so that } \sup\{\|\chi_E f\|_F : f \in \mathcal{K}\} < \varepsilon \text{ whenever } \mu(E) < \delta.$$

For $f \in L_F^*$ we say that f has absolutely continuous norm if $\{f\}$ has equi-absolutely continuous norms.

The following two results deal with the equi-absolute continuity of the norms.

Theorem 1.3.7 *A function $f \in L_F^*$ has absolutely continuous norm if and only if $f \in E_F$.*

Theorem 1.3.8 *If $\mathcal{K} \subset L_F^*$, $\mathcal{K}$ has equi-absolutely continuous norms and $\mathcal{K}$ is relatively compact in the topology of convergence in measure, then $\mathcal{K}$ is relatively (norm) compact in L_F^*.*

1.4 De La Vallée Poussin's theorem revisited

We state and prove the following lemma which can be found in [17, page 62].

Lemma 1.4.1 *Given an N-function F, there is an N-function $H \in \Delta'$ so that $H(H(x)) \leq F(x)$ for large values of x.*

Proof : Write $F = F_1 \circ F_2$, where F_1, F_2 are N-functions and let G_i be the complement of F_i. Let $Q(x) = e^{G_1(x)+G_2(x)}$. The function Q is convex, with $\lim_{x\to\infty} \dfrac{Q(x)}{x} = \infty$. Hence there is an N-function K whose principal part is Q. Clearly $K \in \Delta^2$ and $G_i(x) \leq K(x)$ for large x. So if H is complementary to K, we must have $H \in \Delta'$ and $H(x) \leq F_i(x)$ for large x. Thus $H(H(x)) \leq F_1(F_2(x)) = F(x)$ for large values of x. ∎

Lemma 1.4.2 *If $F \in \Delta_2$ and $\mathcal{K} \subset L_F^*$ then the following statements are equivalent:*

I) The set $\mathcal{K}$ has equi-absolutely continuous norms.

II) The collection $\{F(f) : f \in \mathcal{K}\}$ is uniformly integrable in L^1.

Proof : The implication "(I) ⇒ (II)" follows directly from the fact that

$$\int_E F(f)d\mu = \int F(\chi_E f)d\mu = \mathbf{F}(\chi_E f) \leq \|\chi_E f\|_F$$

whenever $\|\chi_E f\|_F \leq 1$.

Next suppose $\{F(f) : f \in \mathcal{K}\}$ is uniformly integrable. Let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ so that $\frac{1}{2^{n-1}} < \varepsilon$. Since $F \in \Delta_2$, there are $K > 0$, $c > 0$ so that $F(2^n x) \leq KF(x)$ for $x \geq c$. Choose $0 < \delta < \frac{1}{2F(c)}$ so that

$$\sup\left\{\int_E F(f)d\mu : f \in \mathcal{K}\right\} < \frac{1}{2K} \text{ whenever } \mu(E) < \delta.$$

Then for $\mu(E) < \delta$, $f \in \mathcal{K}$ we have

$$\begin{aligned} \int_E F(2^n f)d\mu &\leq \int_E F(c)d\mu + \int_{E\cap[\,|f|\geq c\,]} F(2^n f)d\mu \\ &< \frac{1}{2} + K\int_{E\cap[\,|f|\geq c\,]} F(f)d\mu < 1. \end{aligned}$$

Thus $\|2^n f\chi_E\|_F \leq \int F(2^n f\chi_E)d\mu + 1 < 2$. So $\|f\chi_E\|_F < \frac{1}{2^{n-1}} < \varepsilon$. ∎

From these two lemmas we obtain the following characterization of norm compact subsets of L^1.

Theorem 1.4.3 *A subset $\mathcal{K}$ of $L^1(\mu)$ is relatively compact if and only if there is an N-function $F \in \Delta'$ so that $\mathcal{K}$ is relatively compact in L_F^*.*

Proof : Since the inclusion map $L_F^* \hookrightarrow L^1$ is continuous, necessity follows.

Suppose $\mathcal{K}$ is relatively compact in L^1. Then $\mathcal{K}$ is also relatively weakly compact in L^1 and so by the theorem of De La Vallée Poussin there is an N-function H so that $\sup\{\int H(f)d\mu : f \in \mathcal{K}\} < \infty$. By Lemma (1.4.1) there is an N-function $F \in \Delta'$ with $F(F(x)) \leq H(x)$ for large values of x. Thus $\sup\{\int F(F(f))\, d\mu \mid f \in \mathcal{K}\} < \infty$ and by De La Vallée Poussin's theorem again, we have that $\{F(f) \mid f \in \mathcal{K}\}$ is uniformly integrable in L^1. So by Lemma (1.4.2) $\mathcal{K}$ has equi-absolutely continuous norms in L_F^*. Since $\mathcal{K}$ is relatively compact in L^1, it is also relatively compact in the topology of convergence in measure. Hence $\mathcal{K}$ is relatively compact in L_F^*. ∎

The following results deal with relative weak compactness in L^1 and L_F^*. We begin by mentioning a remarkable theorem of J. Komlós [16].

Theorem 1.4.4 (Komlós) *If (f_n) is bounded in L^1 then there is a subsequence (f_{n_k}) of (f_n) and a function $f \in L^1$ so that each subsequence of (f_{n_k}) has arithmetic means μ-a.e. convergent to f.*

Definition 1.4.1 *A subset S of a Banach space X is a Banach-Saks set if every sequence in S has a subsequence, each subsequence of which has norm convergent arithmetic means. The space X is said to have the Banach-Saks property if every bounded set of X is a Banach-Saks set. Similarily X is said to have the weak Banach-Saks property, if every weakly compact set in X is a Banach-Saks set.*

It is an easy consequence of the Hahn-Banach theorem, that Banach-Saks sets are weakly compact. So we are now ready for the next theorem.

Theorem 1.4.5 *Let $\mathcal{K} \subset L_F^*$. If $\mathcal{K}$ has equi-absolutely continuous norms and it is norm bounded, then $\mathcal{K}$ is a Banach-Saks set in L_F^*. In particular $\mathcal{K}$ is relatively weakly compact in L_F^*.*

Proof : Since $\mathcal{K}$ has equi-absolutely continuous norms, $\mathcal{K} \subset E_F$. Let (f_n) be a sequence in $\mathcal{K}$. Since (f_n) is bounded in L_F^*-norm, it is also bounded in L^1-norm. Hence by Komlós's theorem, there is a subsequence (f_{n_k}) of (f_n) and a function $f \in L^1$ so that any subsequence of (f_{n_k}) has $\mu-$a.e. convergent arithmetic means to f. Let G denote the complement of F. Note that for any measurable E and any $g \in L_G^*$ with $\|g\|_G \leq 1$ we have

$$\begin{aligned}
|\int g\chi_E f\, d\mu| &\leq \int |g\chi_E f| d\mu \\
&\leq \liminf_n \int |g\chi_E \frac{1}{n} \sum_{k=1}^{n} f_{n_k}| d\mu \\
&\leq \sup_n \frac{1}{n} \sum_{k=1}^{n} \int |g\chi_E f_{n_k}| d\mu \\
&\leq \sup_n \frac{1}{n} \sum_{k=1}^{n} \|g\|_G \cdot \|\chi_E f_{n_k}\|_F \\
&\leq \sup\{\|\chi_E h\|_F : h \in \mathcal{K}\}.
\end{aligned}$$

Thus $\|\chi_E f\|_F \leq \sup\{|\int g\chi_E f d\mu| : \|g\|_G \leq 1\} \leq \sup\{\|\chi_E h\|_F : h \in \mathcal{K}\}$. So $f \in L_F^*$ and f has absolutely continuous norm. Let (h_k) be any subsequence of (f_{n_k}) and let $a_n = \frac{1}{n}\sum_{k=1}^{n} h_k$.

We now claim that $a_n \to f$ in L_F^*-norm. Since the inclusion map $L_G^* \hookrightarrow L^1$ is continuous, there is a $K > 0$ so that $\|g\|_1 \leq K\|g\|_G$ for all $g \in L_G^*$. Fix $\varepsilon > 0$ and choose $\delta > 0$ so that $\sup\{\|\chi_A h\|_F : h \in \mathcal{K}\} < \frac{\varepsilon}{3}$ whenever $\mu(A) < \delta$.

By Egorov's theorem, there is a measurable set E with $\mu(\Omega \setminus E) < \delta$ so that $a_n \to f$ uniformly on E. Choose $N \in \mathbb{N}$ so that $\|\chi_E(a_n - f)\|_\infty < \frac{\varepsilon}{3K}$ whenever $n \geq N$. Then for any $g \in L_G^*$ with $\|g\|_G \leq 1$ and $n \geq N$ we have

$$\begin{aligned}
|\int g(a_n - f)d\mu| &\leq \int |g| \cdot |a_n - f| d\mu \\
&= \int_E |g| \cdot |a_n - f| d\mu + \int_{\Omega \setminus E} |g| \cdot |a_n - f| d\mu \\
&\leq \| g \|_1 \cdot \| \chi_E(a_n - f) \|_\infty + \| g \|_G \cdot \| \chi_{\Omega \setminus E}(a_n - f) \|_F \\
&\leq K\|g\|_G \frac{\varepsilon}{3K} + \|g\|_G(\|a_n \chi_{\Omega \setminus E}\|_F + \|f \chi_{\Omega \setminus E}\|_F) \\
&< \frac{\varepsilon}{3} + \| \left(\frac{1}{n} \sum_{k=1}^{n} h_k \right) \chi_{\Omega \setminus E} \|_F + \frac{\varepsilon}{3} \\
&\leq \frac{2\varepsilon}{3} + \frac{1}{n} \sum_{k=1}^{n} \|h_k \chi_{\Omega \setminus E}\| \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

So the claim is established.

Thus $\mathcal{K}$ is a Banach-Saks set in L_F^*. It also follows that $f_{n_k} \to f$ weakly in L_F^* and so $\mathcal{K}$ is relatively weakly compact in L_F^* thanks to the Eberlein-Smulian theorem. ∎

A. Grothendieck has shown that if $1 \leq p < \infty$ and X is a closed subspace of $L^p(\mu)$ contained in $L^\infty(\mu)$, then X is finite dimensional (see [10] and [28, ch. 5]). We generalize this result as follows.

Theorem 1.4.6 *Suppose that* $X \subset L^\infty(\mu)$ *and suppose that* X *is a closed subspace of an Orlicz space* L_F^* *. Then* X *is finite dimensional.*

Proof : Let $i_1 : X \hookrightarrow L^\infty(\mu)$ and $i_2 : L^\infty(\mu) \hookrightarrow L_F^*$ be the natural inclusion maps, with X having the topology inherited from L_F^* . Let (f_n) be a sequence in X and assume that $\| f_n - f \|_F \to 0$ for some $f \in X$. Also assume that $\| f_n - g \|_\infty \to 0$ for some $g \in L^\infty$. The first assumption yields a subsequence (f_{n_k}) of (f_n) with $f_{n_k} \to f$ $\mu - a.e.$. Since $f_n \to g$ uniformly $\mu - a.e.$ we have that $f = g$ $\mu - a.e.$. Thus by the closed graph theorem i_1 is continuous.

Now by Theorem (1.4.5) i_2 is weakly compact and as $L^\infty(\mu)$ has the Dunford-Pettis property, i_2 is completely continuous. Hence $i_2 \circ i_1$ is weakly compact and completely continuous. But $i_2 \circ i_1$ is the identity on X. Now it is not hard to see that the identity on X is compact and hence X is finite dimensional. ∎

We now prove the following stronger version of De La Vallée Poussin's theorem.

Theorem 1.4.7 *A set $\mathcal{K}$ is relatively weakly compact in L^1 if and only if there is $F \in \Delta'$ so that $\mathcal{K}$ is relatively weakly compact in L_F^*.*

Proof : Since the inclusion map $L_F^* \hookrightarrow L^1$ is continuous and thus weak-to-weak continous, necessity follows. So suppose that $\mathcal{K}$ is relatively weakly compact in L^1. By De La Vallée Poussin's theorem, there is an N-function H with $\sup\{\int H(f)d\mu : f \in \mathcal{K}\} < \infty$. By Lemma (1.4.1), there is $F \in \Delta'$ with $F\left(F(x)\right) \leq H(x)$ for large x. So $\sup\{\int F\left(F(f)\right)d\mu : f \in \mathcal{K}\} < \infty$, and by De La Vallée Poussin's theorem once more, we have that $\{F(f) : f \in \mathcal{K}\}$ is relatively weakly compact in L^1. Hence by Lemma (1.4.2), $\mathcal{K}$ has equi-absolutely continuous norms in L_F^*. Since $\mathcal{K}$ is obviously bounded in L_F^*, we then have that $\mathcal{K}$ is relatively weakly compact in L_F^*, thanks to Theorem (1.4.5). ∎

Remark: If $\mathcal{K} \subset L^1$ and if there is an N-function F with its complementary $G \in \Delta_2$ so that $\sup\{\int F(f)d\mu : f \in \mathcal{K}\} < \infty$ then $\mathcal{K}$ is a bounded subset of L^p for some $p > 1$.

Indeed, if $G \in \Delta_2$ then there is $q > 1$ so that $L^q \subset L_G^*$. Let $T : L^q \to L_G^*$ denote the natural inclusion map. Then if $\frac{1}{p} + \frac{1}{q} = 1$ the adjoint operator $T^* : L_F^* \to L^p$ is also a natural inclusion map. Since T is continuous so is T^*. Hence $\mathcal{K}$ bounded in L_F^*, implies that $\mathcal{K}$ is also bounded in L^p.

Chapter 2

ORLICZ SPACES AND THE WEAK BANACH-SAKS PROPERTY

2.1 A weak compactness result reminiscent of the Dunford-Pettis theorem

In this section we deal with a special class of Orlicz spaces, namely those spaces whose generating N-function F satisfies Δ_2 and the function G complementary to F satisfies $lim_{t\to\infty}\frac{G(ct)}{G(t)} = \infty$ for some $c > 0$.[1] This class of spaces has been examined by D. Leung and in [18] they have been shown to satisfy the weak Dunford-Pettis property[2] while they fail the Dunford-Pettis property. This fact shows that such spaces are not isomorphic to $L^1(p)$ for any probability p. Nonetheless they exhibit some striking similarities with L^1 spaces. Some of these similarities are discussed in Chapters 2 and 3. At this point we should mention that V. A. Akimovich has shown in [1] that every reflexive Orlicz space over a probability is isomorphic to a uniformly convex Orlicz space. Combining this result with Kakutani's result in [15] that states that uniformly convex spaces have the Banach-Saks property, one can immediately conclude that reflexive Orlicz spaces have the Banach-Saks property.

Lemma 2.1.1 *Let $\mathcal{K} \subset L_F^*$ where $F \in \Delta_2$. Suppose that $\mathcal{K}$ fails to have equi-absolutely continuous norms. Then there is an $\varepsilon_0 > 0$, a sequence $(f_n) \subset \mathcal{K}$ and a sequence (E_n) of*

[1] The following question remains unresolved. Given an N-function $F \in \Delta_2$ with its complement $G \notin \Delta_2$ does there exist an N-function Φ equivalent to F so that its complement Ψ satisfies $lim_{t\to\infty}\frac{\Psi(ct)}{\Psi(t)} = \infty$ for some $c > 0$?

[2] A Banach lattice X has the weak Dunford-Pettis property if any weakly compact operator from X into any Banach space maps disjoint, weakly null sequences onto norm null sequences.

pairwise disjoint measurable sets, so that $\| \chi_{E_n} f_n \|_F > \varepsilon_0$ *for all positive integers* n.

Proof : Since $\mathcal{K}$ does not have equi-absolutely continuous norms, there is an $\eta_0 > 0$ and sequences $(k_n) \subset \mathcal{K}$, $(A_n) \subset \Sigma$, with $\mu(A_n) < \frac{1}{2^n}$, so that $\| \chi_{A_n} k_n \|_F > \eta_0$ for all positive integers n. For each n let $B_n = \bigcup_{j=n}^{\infty} A_j$. Then $B_n \supset B_{n+1}$. Furthermore

$$\mu(B_n) = \mu(\bigcup_{j=n}^{\infty} A_j) \leq \sum_{j=n}^{\infty} \mu(A_j) \leq \sum_{j=n}^{\infty} \frac{1}{2^j} \to 0$$

as $n \to \infty$, with $\| \chi_{B_n} k_n \|_F \geq \| \chi_{A_n} k_n \|_F > \eta_0$ for all positive integers n. Since $F \in \Delta_2$ we have that each $f \in L_F^*$ has absolutely continuous norm. So if $n_1 = 1$ then there is $n_2 > n_1$ so that $\| \chi_{B_{n_1} \setminus B_{n_2}} k_{n_1} \|_F > \frac{\eta_0}{2}$ (After all $\mu(B_n) \searrow 0$). Let $E_1 = B_{n_1} \setminus B_{n_2}$ and let $f_1 = k_{n_1}$. Now choose $n_3 > n_2$ so that $\| \chi_{B_{n_2} \setminus B_{n_3}} k_{n_2} \|_F > \frac{\eta_0}{2}$. Let $E_2 = B_{n_2} \setminus B_{n_3}$ and let $f_2 = k_{n_2}$. Continue on. The result is now established if we take $\varepsilon_0 = \frac{\eta_0}{2}$. ∎

We next present a 'Rosenthal's Lemma' type of result. (cf. [6, page 82].)

Lemma 2.1.2 *Let* X *be a Banach space. Suppose that* $(x_n) \subset X$ *is weakly null and* $(x_n^*) \subset X^*$ *is weak** *null. Then for each* $\varepsilon > 0$ *there is a subsequence* (n_k) *of the positive integers, so that, for each positive integer* k *we have*

$$\sum_{j \neq k} |< x_{n_j}^*, x_{n_k} >| < \varepsilon \; .$$

Proof : Let $\varepsilon > 0$. Let $n_1 = 1$. Since $x_n^* \to 0$ *weak** there is an infinite subset A_1 of the positive integers so that $\sum_{j \in A_1} |< x_j^*, x_{n_1} >| < \frac{\varepsilon}{2}$. Since $x_n \to 0$ weakly and since A_1 is infinite, we can find $n_2 > n_1$ with $n_2 \in A_1$, so that $|< x_{n_1}^*, x_{n_2} >| < \frac{\varepsilon}{2}$. Similarly there is an infinite subset A_2 of A_1 so that $\sum_{j \in A_2} |< x_j^*, x_{n_2} >| < \frac{\varepsilon}{2}$. Again choose $n_3 > n_2$ with $n_3 \in A_2$ so that $|< x_{n_1}^*, x_{n_3} >| < \frac{\varepsilon}{4}$ and $|< x_{n_2}^*, x_{n_3} >| < \frac{\varepsilon}{4}$. There is an infinite subset A_3 of A_2 so that $\sum_{j \in A_3} |< x_j^*, x_{n_3} >| < \frac{\varepsilon}{2}$. Choose $n_4 > n_3$ with $n_4 \in A_3$ so that $|< x_{n_i}^*, x_{n_4} >| < \frac{\varepsilon}{6}$ for $i = 1 \ldots 3$. Continue inductively to construct a sequence of infinite subsets of the positive integers, $A_1 \supset A_2 \cdots \supset A_k \supset \cdots$ and a sequence $n_1 < n_2 < \cdots$ of positive integers with

$$(i) \qquad n_{k+1} \in A_k \ \ for \ \ all \ \ k.$$

$$(ii) \qquad \sum_{j \in A_k} |< x_j^*, x_{n_{k+1}} >| < \frac{\varepsilon}{2} \ \ for \ \ all \ \ k.$$

$$(iii) \qquad |< x_{n_j}^*, x_{n_{k+1}} >| < \frac{\varepsilon}{2k} \ \ for \ \ all \ \ k \ \ and \ \ for \ \ j = 1, 2, \ldots, k.$$

Now for fixed positive integer k we have

$$\begin{aligned} \sum_{j \neq k} |< x_{n_j}^*, x_{n_k} >| &= \sum_{j=1}^{k-1} |< x_{n_j}^*, x_{n_k} >| + \sum_{j=k+1}^{\infty} |< x_{n_j}^*, x_{n_k} >| \\ &< \frac{\varepsilon}{2(k-1)}(k-1) + \sum_{j \in A_k} |< x_{n_j}^*, x_{n_k} >| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

And so we are done. ∎

Now we are ready for the main result of this section.

Theorem 2.1.3 *Suppose that $F \in \Delta_2$ and that its complement G satisfies*

$$\lim_{t \to \infty} \frac{G(ct)}{G(t)} = \infty \ \ for \ \ some \ \ c > 0.$$

Then any weakly null sequence in L_F^* has equi-absolutely continuous norms.

Proof : Suppose not. Then there is a weakly null sequence $(f_n) \subset L_F^*$ that fails to have equi-absolutely continuous norms. Using Lemma (2.1.1) we may assume that there is an $\varepsilon_0 > 0$ and a sequence (E_n) of pairwise disjoint measurable sets so that $\| \chi_{E_n} f_n \|_F > \varepsilon_0$ for all positive integers n. Now choose a sequence $(g_n) \subset L_G$ so that each g_n is supported on E_n with $\int G(g_n)\, d\mu \ \leq \ 1$ and so that $| \int g_n f_n d\mu | > \ \varepsilon_0$. For a fixed $f \in L_F^*$ Hölder's Inequality yields

$$| \int f g_n d\mu | = | \int \chi_{E_n} f g_n d\mu | \leq \| \chi_{E_n} f \|_F \ \cdot \| g_n \|_G \ .$$

But since (E_n) are pairwise disjoint and μ is finite we have that $\mu(E_n) \to 0$. Furthermore since $F \in \Delta_2$ and $f \in L_F^*$, f has absolutely continuous norm. Thus $\| \chi_{E_n} f \|_F \to 0$. As

(g_n) is norm bounded, we can conclude that $\| \chi_{E_n} f \|_F \cdot \| g_n \|_G \to 0$ and so $\int f g_n d\mu \to 0$. Hence (g_n) is weak* null. By Lemma (2.1.2) there is a subsequence (n_k) of the positive integers so that for each k we have $\sum_{j \neq k} | \int g_{n_j} f_{n_k} d\mu | < \frac{\varepsilon_0}{2}$.

We now claim that $\int G(\frac{g_n}{c}) d\mu \to 0$. Fix $\varepsilon > 0$. Since $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ then $\lim_{t\to\infty} \frac{G(t/c)}{G(t)} = 0$. Choose $t_0 > 0$ so that $\frac{G(t/c)}{G(t)} < \frac{\varepsilon}{2}$ whenever $t \geq t_0$. Since $\mu(E_n) \to 0$, there is a positive integer N so that $\mu(E_n) < \frac{\varepsilon}{2G(t_0/c)}$ whenever $n \geq N$. Hence if $n \geq N$ we have

$$\begin{aligned}
\int G(g_n/c)d\mu &= \int_{[\,|g_n|<t_0\,]} G(g_n/c)d\mu + \int_{[\,|g_n|\geq t_0\,]} G(g_n/c)d\mu \\
&\leq G(t_0/c)\mu(E_n) + \int \frac{\varepsilon}{2} G(g_n)d\mu \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.
\end{aligned}$$

So the claim is established.

Now choose a subsequence (n_{k_m}) of (n_k) so that

$$\sum_{m=1}^{\infty} \int G(\frac{g_{n_{k_m}}}{c})d\mu < \infty\,.$$

Let $g = \sum_{m=1}^{\infty} g_{n_{k_m}}$. Then g is well defined and $g \in L_G^*$, since $\int G(g/c)d\mu < \infty$. Since (f_n) is weakly null, we must have $\int g f_{n_{k_m}} d\mu \to 0$ *as* $m \to \infty$. But for each positive integer m we have

$$\begin{aligned}
|\int g f_{n_{k_m}} d\mu\,| &= |\int (\sum_{j=1}^{\infty} g_{n_{k_j}}) f_{n_{k_m}} d\mu\,| \\
&\geq |\int g_{n_{k_m}} f_{n_{k_m}} d\mu\,| - \sum_{j\neq m} |\int g_{n_{k_j}} f_{n_{k_m}} d\mu\,| \\
&\geq |\int g_{n_{k_m}} f_{n_{k_m}} d\mu\,| - \sum_{j\neq k_m} |\int g_{n_j} f_{n_{k_m}} d\mu\,| \\
&> \varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2}\,,
\end{aligned}$$

which is a contradiction. ∎

As a corollary to the theorem above, we get the following result that resembles the Dunford-Pettis theorem for L^1.

Corollary 2.1.4 *Let $F \in \Delta_2$ and suppose that its complement G satisfies*

$$\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty \ for\ some\ c > 0\,.$$

Then a bounded set $\mathcal{K} \subset L_F^*$ is relatively weakly compact if and only if $\mathcal{K}$ has equi-absolutely continuous norms.

Proof : Suppose that $\mathcal{K} \subset L_F^*$ is relatively weakly compact. If $\mathcal{K}$ fails to have equi-absolutely continuous norms then there is an $\varepsilon_0 > 0$, a sequence $(f_n) \subset \mathcal{K}$ and a sequence (E_n) of measurable sets with $\mu(E_n) \to 0$ so that $\| \chi_{E_n} f_n \|_F > \varepsilon_0$, for each positive integer n. By the Eberlein-Smulian theorem, there is an $f \in L_F^*$ and a subsequence (f_{n_k}) of (f_n) so that $f_{n_k} \to f$ weakly in L_F^*. So by Theorem (2.1.3), $(f_{n_k} - f)$ has equi-absolutely continuous norms. Thus $\| \chi_{E_{n_k}}(f_{n_k} - f) \|_F \to 0$ as $k \to \infty$. As $F \in \Delta_2$ and $f \in L_F^*$, f has absolutely continuous norm. Hence $\| \chi_{E_{n_k}} f \|_F \to 0$ as $k \to \infty$. But

$$\varepsilon_0 < \| \chi_{E_{n_k}} f_{n_k} \|_F \le \| \chi_{E_{n_k}} f \|_F + \| \chi_{E_{n_k}} (f_{n_k} - f) \|_F$$

which is a contradiction.

The converse is just Theorem (1.4.5). ∎

Corollary 2.1.5 *Under the hypothesis of Corollary (2.1.4), L_F^* has the weak Banach-Saks property.*

Proof : It follows directly from Corollary (2.1.4) and Theorem (1.4.5). ∎

2.2 An application in convex function theory

Recall that an N-function G satisfies the Δ_3 condition if there is $c > 0$ so that $tG(t) \le G(ct)$ for large values of t. If $G \in \Delta_3$ then its complement $F \in \Delta_2$ [17, pages 29–30]. Furthermore it is clear that $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$. Also note that if $G \in \Delta^2$ then $G \in \Delta_3$.

In [17, page 30] the following question is posed: Given an N-function $F \in \Delta'$ is it possible to find an N-function H, equivalent to F so that for some $K > 0$

$$H(xy) \le K \cdot H(x) \cdot H(y) \quad \forall\, x, y \in \mathbb{R}\,?$$

The following theorem answers this question in the negative.

Theorem 2.2.1 *Suppose that $G \in \Delta^2$ and let F denote the complement of G. Then there is no N-function H equivalent to F which satisfies the following condition:*
There is a $K > 0$ so that $H(t_1 \cdot t_2) \leq K \cdot H(t_1) \cdot H(t_2)$ for all real t_1 and t_2.

Proof : Suppose that such an H existed. Let μ denote Lebesgue measure on the interval $[0,1]$. Since μ is non-atomic, we can find a sequence (E_n) of pairwise disjoint measurable sets, each of which has positive measure. For each positive integer n, let $h_n = H^{-1}(\frac{1}{\mu(E_n)})\chi_{E_n}$. Then $h_n \in L_H^*$ with $\int H(h_n)d\mu = 1$ for all positive integers n. It follows from Lemma (1.4.2) that no subsequence of (h_n) has equi-absolutely continuous norms.

We now claim that (h_n) is weakly null. Let (h_{n_k}) be any subsequence of (h_n). Then for any positive integer N we have

$$\begin{aligned}
\int H(\frac{1}{N}\sum_{k=1}^{N} h_{n_k})d\mu &\leq K \cdot H(\frac{1}{N}) \cdot \int H(\sum_{k=1}^{N} h_{n_k})d\mu \\
&= K \cdot H(\frac{1}{N}) \cdot \sum_{k=1}^{N} \int H(h_{n_k})d\mu \\
&= K \cdot H(\frac{1}{N}) \cdot \sum_{k=1}^{N} \frac{1}{\mu(E_{n_k})}\mu(E_{n_k}) \\
&= K \cdot H(\frac{1}{N}) \cdot N \ .
\end{aligned}$$

Since H is an N-function, $\lim_{t\to 0} \frac{H(t)}{t} = 0$. Thus $\lim_{N\to\infty} K \cdot H(\frac{1}{N}) \cdot N = 0$. But since $H \in \Delta'$ then $H \in \Delta_2$. So $\| \frac{1}{N}\sum_{k=1}^{N} h_{n_k} \|_H \to 0$. To summarize, every subsequence of (h_n) has norm null arithmetic means and so (h_n) is weakly null as we claimed. Now since F is equivalent to H, there are constants $\lambda_1 > 0$ and $\lambda_2 > 0$ so that

$$\lambda_1 \| f \|_F \leq \| f \|_H \leq \lambda_2 \| f \|_F \quad for\ all\ f \in L_H^*(= L_F^*) \ .$$

By Theorem(2.1.3), (h_n) has equi-absolutely continuous F-norms and thus, by the inequality above, (h_n) also has equi-absolutely continuous H-norms.

But this is clearly a contradiction. ∎

Remark: The same result can be obtained from the work of T. Ando in [2]. Specifically it follows directly from [2, Theorem 1], that *given $F \in \Delta_2$, a subset $\mathcal{K}$ of L_F^* is relatively weakly compact, if and only if*

$$\lim_{t\to 0}(\sup\{\frac{\mathbf{F}(tf)}{t} : f \in \mathcal{K}\}) = 0$$

With this fact in hand, we can easily prove the following theorem.

Theorem 2.2.2 *Let F be an N-function satisfying the Δ' condition for all real x, y. That is there is $K > 0$ so that $F(xy) \le K \cdot F(x) \cdot F(y)$ for all $x, y \in \mathbb{R}$. Then L_F^* is reflexive.*

Proof: Since $F \in \Delta'$ then $f \in \Delta_2$. Furthermore

$$\begin{aligned}
\lim_{t\to 0}(\sup\{\frac{\mathbf{F}(tf)}{t} : f \in B_{L_F^*}\}) &= \lim_{t\to 0}(\sup\{\frac{\int_\Omega F(tf(\omega))d\mu(\omega)}{t} : f \in B_{L_F^*}\}) \\
&\le \lim_{t\to 0}(\sup\{\frac{\int_\Omega K \cdot F(t) \cdot F(f(\omega))d\mu(\omega)}{t} : f \in B_{L_F^*}\}) \\
&= \lim_{t\to 0}\frac{K \cdot F(t)}{t} = 0 \,.
\end{aligned}$$

Thus $B_{L_F^*}$ is relatively weakly compact and so L_F^* is reflexive. ∎

Now it is easy to see that given any N-function $F \in \Delta'$ so that its complement $G \notin \Delta_2$, then there is no N-function H equivalent to F so that, H satisfies Δ' for all real x, y.

Chapter 3

REFLEXIVE SUBSPACES OF NON-REFLEXIVE ORLICZ SPACES.

3.1 Subspaces containing complemented l_1

In this section we derive a theorem similar to the one of Kadec and Pelczýnski, about L^1 in [13]. The proofs are modeled after the ones in [6, pages 94-98].

Lemma 3.1.1 *Let (f_n) be a normalized disjointly supported sequence in L_F^*, where $F \in \Delta_2$ and its complement G satisfies $\lim_{x\to\infty} \frac{G(cx)}{G(x)} = \infty$, for some $c > 0$. Then there is a subsequence (f_{n_k}) of (f_n) so that*

i. *(f_{n_k}) is equivalent to l_1's unit vector basis.*

ii. *The closed linear span of (f_{n_k}) is complemented in L_F^* by means of a projection of norm less than or equal to $4c$.*

iii. *The coefficient functionals (ϕ_k) extend to all of the dual of L_F^* and $\|\phi_k\| \leq 4$ for all positive integers k.*

Proof : Let E_n denote the support of f_n. For each positive integer n choose $g_n \in L_G$ with $\int G(g_n)d\mu \leq 1$ so that $\int g_n f_n d\mu \geq \frac{1}{2}$. There is no harm in assuming that each g_n is also supported on E_n.

Claim that $\int G(g_n/c)d\mu \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$. Since $\lim_{x\to\infty} \frac{G(cx)}{G(x)} = \infty$ then $\lim_{x\to\infty} \frac{G(x/c)}{G(x)} = 0$. So we can choose $x_0 > 0$ so that $\frac{G(x/c)}{G(x)} < \frac{\varepsilon}{2}$ whenever $x \geq x_0$. Since the E_n's are pairwise disjoint and μ is a probability, we have that $\mu(E_n) \to 0$ as $n \to \infty$.

So there is a positive integer N so that $\mu(E_n) < \frac{\varepsilon}{2G(x_0/c)}$ whenever $n \geq N$. So for $n \geq N$ we have

$$\begin{aligned}\int G(g_n/c)d\mu &= \int_{[\,|g_n|<x_0\,]} G(g_n/c)d\mu + \int_{[\,|g_n|\geq x_0\,]} G(g_n/c)d\mu \\ &\leq G(x_0/c)\mu(E_n) + \frac{\varepsilon}{2}\int G(g_n)d\mu \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and so the claim is established.

Now choose a subsequence (n_k) of the positive integers so that $\sum_{k=1}^{\infty}\int G(\frac{g_{n_k}}{c})d\mu \leq 1$. For any sequence of signs $\sigma = (\varepsilon_k)$ define $g_\sigma = \sum_{k=1}^{\infty}\varepsilon_k g_{n_k}$. Since the g_{n_k}'s are disjointly supported, g_σ is well defined. Furthermore

$$\begin{aligned}\int G(\frac{g_\sigma}{c})d\mu &= \sum_{k=1}^{\infty}\int_{E_{n_k}} G(\frac{g_\sigma}{c})d\mu \\ &= \sum_{k=1}^{\infty}\int_{E_{n_k}} G(\frac{\varepsilon_k g_{n_k}}{c})d\mu \\ &= \sum_{k=1}^{\infty}\int_{E_{n_k}} G(\frac{g_{n_k}}{c})d\mu \\ &\leq 1\,.\end{aligned}$$

So $g_\sigma \in L_G^*$. Recall that the norm of g_σ in L_G^*, is given by $\|g_\sigma\|_G = \inf\{\frac{1}{k}(1+\int G(kg_\sigma)d\mu) : k > 0\}$ and so it is easy to see that $\|g_\sigma\|_G$ remains constant as σ varies. Denote this constant by M and observe that

$$M = \|g_\sigma\|_G \leq c(1+\int G(\frac{g_\sigma}{c})d\mu) = c(1+\sum_{k=1}^{\infty}\int G(\frac{g_{n_k}}{c})d\mu) \leq 2c\,.$$

Now for $(a_k) \in l_1$ let $\sigma = (sign(a_k))$. Then

$$\begin{aligned}\|\sum_{k=1}^{\infty} a_k f_{n_k}\|_F &\geq \frac{1}{\|g_\sigma\|_G}\int (g_\sigma \sum_{k=1}^{\infty} a_k f_{n_k})d\mu \\ &= \frac{1}{M}\int (\sum_{k=1}^{\infty} |a_k| g_{n_k} f_{n_k})d\mu\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{M}\sum_{k=1}^{\infty}|a_k|\int g_{n_k}f_{n_k}d\mu \\
&\geq \frac{1}{2M}\sum_{k=1}^{\infty}|a_k| \ .
\end{aligned}
$$

Hence (i) is established.

Now define for each k, a functional ϕ_k on all of L_F^* by

$$\phi_k(f) = \frac{1}{\int g_{n_k}f_{n_k}d\mu}\cdot\int g_{n_k}f d\mu$$

and define $P : L_F^* \to L_F^*$ by

$$P(f) = \sum_{k=1}^{\infty}\phi_k(f)f_{n_k}.$$

Then for $k = 1, 2, \ldots$

$$\|\phi_k\| \leq 2\|g_{n_k}\|_G \leq 2\cdot(1+\mathbf{G}(g_{n_k})) \leq 4 \ .$$

Furthermore P is a projection of L_F^* onto the closed linear span of (f_{n_k}) with

$$
\begin{aligned}
\|P\| &= \sup_{\|f\|_F\leq 1}\|\sum_{k=1}^{\infty}\frac{\int g_{n_k}fd\mu}{\int g_{n_k}f_{n_k}d\mu}\cdot f_{n_k}\|_F \\
&\leq 2\sup_{\|f\|_F\leq 1}\sum_{k=1}^{\infty}\int|g_{n_k}f|d\mu \\
&\leq 2\sup_{\|f\|_F\leq 1}\|(\sum_{k=1}^{\infty}|g_{n_k}|)\|_G\cdot\|f\|_F \\
&= 2M \\
&\leq 4c.
\end{aligned}
$$

And so our proof is complete. ∎

We state now the following result in form of a lemma. Its proof can be found in [6, page 50].

Lemma 3.1.2 *Let (z_n) be a basic sequence in the Banach space X with coefficient functionals (z_n^*). Suppose that there is a bounded linear projection $P : X \to X$ onto the closed* *linear span* $[z_n]$ *of* (z_n)*. If* (y_n) *is any sequence in* X *for which*

$$\sum_{n=1}^{\infty} \|P\| \cdot \|z_n^*\| \cdot \|z_n - y_n\| < 1,$$

then (y_n) *is a basic sequence equivalent to* (z_n) *and the closed linear span* $[y_n]$ *of* (y_n) *is also complemented in* X*.*

Lemma 3.1.3 *Let* (f_n) *be a sequence in* L_F^* *where* $F \in \Delta_2$ *and its complement* G *satisfies* $\lim_{x\to\infty} \frac{G(cx)}{G(x)} = \infty$ *for some* $c > 0$*. Suppose that for each* $\varepsilon > 0$ *there is a positive integer* n_ε *so that* $\mu([\,|f_{n_\varepsilon}| \geq \varepsilon \|f_{n_\varepsilon}\|_F\,]) < \varepsilon$*. Then there is a subsequence* (r_n) *of* (f_n) *so that* $(\frac{r_n}{\|r_n\|_F})$ *is equivalent to* l_1*'s unit vector basis. Furthermore the closed linear span* $[r_n]$ *of* (r_n) *is complemented in* L_F^**.*

Proof : First observe that if $f \in L_F^*$, $E = [|f| \geq \varepsilon \|f\|_F]$ and K is the norm of the inclusion map $L_G^* \hookrightarrow L^1$ then

$$\begin{aligned}
\|\chi_E \frac{f}{\|f\|_F}\|_F &\geq \|\frac{f}{\|f\|_F}\|_F - \|\chi_{E^c}\frac{f}{\|f\|_F}\|_F \\
&= 1 - \frac{1}{\|f\|_F}\sup\{|\int g\chi_{E^c} f d\mu| : g \in L_G \textit{ and } \mathbf{G}(g) \leq 1\} \\
&\geq 1 - \frac{1}{\|f\|_F}\sup\{\|g\|_1 \cdot \|\chi_{E^c} f\|_\infty : g \in L_G \textit{ and } \mathbf{G}(g) \leq 1\} \\
&\geq 1 - \frac{K}{\|f\|_F}\|\chi_{E^c} f\|_\infty \\
&\geq 1 - \frac{K}{\|f\|_F}\|f\|_F \cdot \varepsilon \\
&= 1 - K\varepsilon\ .
\end{aligned}$$

So using the hypothesis there is a measurable set E_1 and a positive integer n_1 so that

$$\mu(E_1) < \frac{1}{16c \cdot 4^2 K} \textit{ and } \|\chi_{E_1}\frac{f_{n_1}}{\|f_{n_1}\|_F}\|_F \geq 1 - \frac{1}{16c\cdot 4^2}\ .$$

Since $F \in \Delta_2$ then each $f \in L_F^*$ has an absolutely continuous norm. This fact together with the hypothesis again, yields a measurable E_2 and a positive integer $n_2 > n_1$ so that

$$\mu(E_2) < \frac{1}{16c \cdot 4^3 K},$$

$$\|\chi_{E_2}\frac{f_{n_2}}{\|f_{n_2}\|_F}\|_F > 1 - \frac{1}{16c\cdot 4^3}$$

and

$$\|\chi_{E_2}\frac{f_{n_1}}{\|f_{n_1}\|_F}\|_F < \frac{1}{16c\cdot 4^3}\ .$$

Continue inductively to construct a subsequence (g_n) of (f_n) and a sequence of measurable sets (E_n) so that

$$\mu(E_n) < \frac{1}{16c\cdot 4^{n+1}K},$$

$$\|\chi_{E_n}\frac{g_n}{\|g_n\|_F}\|_F > 1 - \frac{1}{16c\cdot 4^{n+1}}$$

and

$$\sum_{k=1}^{n-1}\|\chi_{E_n}\frac{g_k}{\|g_k\|_F}\|_F < \frac{1}{16c\cdot 4^{n+1}}\ .$$

Now let

$$A_n = E_n \setminus \bigcup_{k=n+1}^{\infty} E_k \quad and \quad h_n = \frac{g_n}{\|g_n\|_F}\chi_{A_n}\ .$$

Then

$$\begin{aligned}
\|\frac{g_n}{\|g_n\|_F} - h_n\|_F &= \|\chi_{A_n^c}\frac{g_n}{\|g_n\|_F}\|_F \\
&\le \|\chi_{E_n^c}\frac{g_n}{\|g_n\|_F}\|_F + \|\chi_{E_n\setminus A_n}\frac{g_n}{\|g_n\|_F}\|_F \\
&\le \frac{1}{16c\cdot 4^{n+1}} + \|\chi_{\bigcup_{k=n+1}^{\infty}E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\le \frac{1}{16c\cdot 4^{n+1}} + \|\sum_{k=n+1}^{\infty}\chi_{E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\le \frac{1}{16c\cdot 4^{n+1}} + \sum_{k=n+1}^{\infty}\|\chi_{E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\le \frac{1}{16c\cdot 4^{n+1}} + \sum_{k=n+1}^{\infty}\frac{1}{16c\cdot 4^{k+1}} \\
&< \frac{1}{16c\cdot 4^{n}}\ .
\end{aligned}$$

Thus

$$1 \;\ge\; \|h_n\|_F$$

$$
\begin{aligned}
&= \|\chi_{A_n}\frac{g_n}{\|g_n\|_F}\|_F \\
&\geq \|\chi_{E_n}\frac{g_n}{\|g_n\|_F}\|_F - \|\chi_{\bigcup_{k=n+1}^{\infty}E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\geq 1 - \frac{1}{16c\cdot 4^{n+1}} - \sum_{k=n+1}^{\infty}\|\chi_{E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\geq 1 - \frac{1}{16c\cdot 4^{n+1}} - \sum_{k=n+1}^{\infty}\frac{1}{16c\cdot 4^{k+1}} \\
&> 1 - \frac{1}{16c\cdot 4^{n}}\ .
\end{aligned}
$$

And so

$$
\begin{aligned}
\|\frac{g_n}{\|g_n\|_F} - \frac{h_n}{\|h_n\|_F}\|_F &\leq \|\frac{g_n}{\|g_n\|_F} - h_n\|_F + \|h_n - \frac{h_n}{\|h_n\|_F}\|_F \\
&\leq \frac{1}{16c\cdot 4^n} + (1 - \|h_n\|_F) \\
&\leq \frac{1}{16c\cdot 4^n} + (1 - 1 + \frac{1}{16c\cdot 4^n}) \\
&= \frac{2}{16c\cdot 4^n}\ .
\end{aligned}
$$

By Lemma (3.1.1), there is a subsequence (n_k) of the positive integers so that

- $(\frac{h_{n_k}}{\|h_{n_k}\|_F})$ is equivalent to l_1's unit vector basis.
- The closed linear span $[h_{n_k}]$ of (h_{n_k}) is complemented in L_F^* by means of a projection P, of norm less than or equal to $4c$.
- The coefficient functionals ϕ_k extend to all of L_G^* with $\|\phi_k\|_G \leq 4$ for all k.

So we have that if $r_k = g_{n_k}$ then

$$
\begin{aligned}
\sum_{k=1}^{\infty}\|P\|\cdot\|\phi_k\|_G\cdot\|\frac{r_k}{\|r_k\|_F} - \frac{h_{n_k}}{\|h_{n_k}\|_F}\|_F &\leq 16c\cdot\sum_{k=1}^{\infty}\|\frac{g_{n_k}}{\|g_{n_k}\|_F} - \frac{h_{n_k}}{\|h_{n_k}\|_F}\|_F \\
&\leq 16c\cdot\sum_{n=1}^{\infty}\|\frac{g_n}{\|g_n\|_F} - \frac{h_n}{\|h_n\|_F}\|_F \\
&\leq 16c\cdot\sum_{n=1}^{\infty}\frac{2}{16c\cdot 4^n} \\
&= \sum_{n=1}^{\infty}\frac{2}{4^n} \\
&< 1\ .
\end{aligned}
$$

Hence the result is established by an appeal to Lemma (3.1.2). ∎

Theorem 3.1.4 *Let $F \in \Delta_2$ with its complement G satisfying*

$$\lim_{x\to\infty} \frac{G(cx)}{G(x)} = \infty \quad for \quad some \quad c > 0.$$

If X is any non-reflexive subspace of L_F^* then X contains an isomorphic copy of l_1 that is complemented in L_F^*.

Proof : Since X is not reflexive, then the ball B_X of X is not relatively weakly compact. Hence by Theorem (2.1.4), B_X does not have equi-absolutely continuous norms. So by Lemma (1.4.2), the set $\{F(f) \,:\, f \in B_X\}$ is not uniformly integrable in L^1. Thus there is a $\delta > 0$ so that

$$\lim_{a\to\infty} \sup\{\int_{[\,|f|\geq a\,]} F(f)d\mu \,;\; f \in B_X\} = \delta \;.$$

Keeping in mind that the above limit is actually an infimum we can find an increasing sequence (a_n) of positive reals, with $a_n \to \infty$ as $n \to \infty$ so that

$$\delta \leq \sup\{\int_{[\,|f|\geq a_n\,]} F(f)d\mu \,;\; f \in B_X\} < \delta + \frac{1}{n} \;,$$

for each positive integer n. It follows then, that there is a sequence (f_n) in B_X so that

$$\delta - \frac{1}{n} < \int_{[\,|f_n|\geq a_n\,]} F(f_n)d\mu < \delta + \frac{1}{n}$$

for all positive integers n. Now let $g_n = f_n\chi_{[\,|f_n|\geq a_n\,]}$ and $h_n = f_n - g_n$. Observe that for each $\varepsilon > 0$ we have

$$\begin{aligned}
\mu([\,|g_n| \geq \varepsilon\|g_n\|_F\,]) &\leq \mu([\,|g_n| > 0\,]) \\
&\leq \mu([\,|f_n| \geq a_n\,]) \\
&\leq \frac{1}{a_n}\int_{[\,|f_n|\geq a_n\,]} |f_n|d\mu \\
&\leq \frac{1}{a_n}\int_{[\,|f_n|\geq a_n\,]} F(f_n)d\mu \\
&\leq \frac{1}{a_n} \;,
\end{aligned}$$

provided that n is large enough. Since $\frac{1}{a_n} \to 0$ as $n \to \infty$ then $\mu([\,|g_n| \geq \varepsilon \|g_n\|_F\,]) < \varepsilon$ for even larger n. So by Lemma (3.1.3), (g_n) has a subsequence that spans a complemented l_1 in L_F^*.

We now show that (h_n) has equi-absolutely continuous norms. Note that if $m \leq n$ then $[\,|h_m| \geq a_n\,] = \emptyset$ while if $m > n$ then

$$\begin{aligned}
\int_{[\,|h_m| \geq a_n\,]} F(h_m) d\mu &= \int_{[\,|f_m| < a_m\,] \cap [\,|f_m| \geq a_n\,]} F(f_m) d\mu \\
&= \int_{[\,|f_m| \geq a_n\,]} F(f_m) d\mu - \int_{[\,|f_m| \geq a_m\,]} F(f_m) d\mu \\
&\leq \sup\{\int_{[\,|f| \geq a_n\,]} F(f) d\mu \;:\; f \in B_X\} - \delta + \frac{1}{m} \\
&\leq \delta + \frac{1}{n} - \delta + \frac{1}{n} \\
&= \frac{2}{n}\,.
\end{aligned}$$

So for each positive integer n we have

$$\sup_m \int_{[\,|h_m| \geq a_n\,]} F(h_m) d\mu = \sup_{m > n} \int_{[\,|h_m| \geq a_n\,]} F(h_m) d\mu \leq \frac{2}{n}\,.$$

It follows then that $\{F(h_m) : m \geq 1\}$ is uniformly integrable in L^1 and so by Lemma (1.4.2), (h_n) has equi-absolutely continuous norms as we claimed. Hence by Corollary (2.1.4), (h_n) is relatively weakly compact in L_F^*. So by passing to appropriate subsequences, we can assume that (g_n) spans a complemented l_1 in L_F^* and (h_n) is weakly convergent in L_F^*. Thus $(h_{2n} - h_{2n+1})$ is weakly null. So by Mazur's theorem, there is an increasing sequence (n_k) of positive integers and a sequence (a_k) of non-negative reals so that

- $\sum_{j=n_k+1}^{n_{k+1}} a_j = 1$.
- The sequence (w_k) defined by $w_k = \sum_{j=n_k+1}^{n_{k+1}} a_j (h_{2j} - h_{2j+1})$ is norm-null in L_F^*.

Let

$$u_k = \sum_{j=n_k+1}^{n_{k+1}} a_j (f_{2j} - f_{2j+1})$$

and

$$v_k = \sum_{j=n_k+1}^{n_{k+1}} a_j(g_{2j} - g_{2j+1}) \ .$$

Then $u_k = v_k + w_k$ and $\|u_k - v_k\|_F = \|w_k\|_F \to 0$ as $k \to \infty$. By selection, $(\frac{g_n}{\|g_n\|_F})$ was equivalent to l_1's unit vector basis with complemented span in L_F^*. As $\|g_n\|_F \geq \int F(g_n)d\mu \geq \delta - \frac{1}{n}$, (g_n) itself is equivalent to l_1's unit vector basis. A little thought convinces us that this is also the case with (v_k), with the closed linear span of (v_k) still complemented in L_F^* of course. By passing to a subsequence to ensure that $\|u_k - v_k\|_F$ converges to zero fast enough to apply Lemma (3.1.2), the result is finished. ∎

3.2 Some facts about Banach Spaces with type

In this section, we denote by (r_n), the sequence of Rademacher functions. Recall that for a positive integer n, $r_n : [0,1] \to \{-1, 1\}$ is defined by

- $r_n(1) = -1$.
- $r_n(t) = (-1)^{(i-1)}$ for $t \in [\frac{i-1}{2^n}, \frac{i}{2^n})$, where $i = 1, \ldots, 2^n$.

Definition 3.2.1 *A Banach space X is said to have type p, for some $1 < p \leq 2$, if there is a constant K so that*

$$(\int_0^1 \|\sum_{i=1}^n r_i(t)x_i\|^p dt)^{\frac{1}{p}} \leq K(\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}} \ ,$$

for any $x_1, \ldots, x_n \in X$.

The following result allows some computational freedom :

Theorem 3.2.1 (Kahane's inequality) *A Banach space X has type $1 < p \leq 2$ if and only if for each $1 \leq q < \infty$ there is a constant $K_q > 0$ such that*

$$(\int_0^1 \|\sum_{i=1}^n r_i(t)x_i\|^q dt)^{\frac{1}{q}} \leq K_q(\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}}$$

for any $x_1, \ldots, x_n \in X$.

It turns out that type's presence in a Banach space, is ultimately connected with the space's finite dimensional structure. To be more specific, we need the following notion.

Definition 3.2.2 *Let $\lambda \geq 1$ and X be a Banach space. We say that X contains l_1^n's λ-uniformly if for each positive integer n there is an isomorphism $T : l_1^n \to X$ so that $\|T\| \cdot \|T^{-1}\| \leq \lambda$.*

It is easy to see from the definition above that X contains l_1^n's λ-uniformly if and only if for each positive integer n, $\exists\, x_1, \ldots, x_n \in B_X$ such that

$$\|\sum_{i=1}^{n} a_i x_i\| \geq \frac{1}{\lambda} \sum_{i=1}^{n} |a_i| \ ,$$

for all choices of scalars $a_1, \ldots, a_n$.

Theorem 3.2.2 (Pisier) *The following are equivalent for a Banach space X :*

1. *For each $\lambda > 1$, X does not contain l_1^n's λ-uniformly.*
2. *For some $\lambda > 1$, X does not contain l_1^n's λ-uniformly.*
3. *The space X has type p for some $1 < p \leq 2$.*

For a proof of this theorem as well as a more detailed account and bibliography, the reader should consult [26] and [25, pages 31-40].

3.3 Subspaces of L_F^* that have type

The work of Kadec and Pelczýnski in [13], finds its natural continuation in the work of Rosenthal. In [27], Rosenthal shows that a subspace of L^1 is reflexive if and only if it has non-trivial type. In this section, we follow his lead, to show that the same fact holds true for the special class of Orlicz spaces, we have been considering. The following result, mentioned in the form of a lemma, is due to Dor and Kauffman (see appendix and [8]).

Lemma 3.3.1 *Suppose* $f_1, \ldots, f_n \in B_{L^1(\mu)}$ *satisfy*

$$\| \sum_{i=1}^{n} a_i f_i \|_1 \geq \theta \sum_{i=1}^{n} |a_i| \ .$$

for any $a_1, \ldots, a_n$*, where* $0 < \theta < 1$*.*

Then there exist pairwise disjoint measurable sets $A_1, \ldots, A_n$ *such that*

$$\int_{A_i} |f_i| d\mu \geq \theta^2 \ .$$

We now adapt that lemma to our purposes.

Lemma 3.3.2 *Suppose* $f_1, \ldots, f_n \in B_{L_F^*(\mu)}$ *satisfy*

$$\| \sum_{i=1}^{n} a_i f_i \|_F \geq \theta \sum_{i=1}^{n} |a_i| \ ,$$

for any $a_1, \ldots, a_n$*, where* $0 < \theta < 1$*. Then there exist pairwise disjoint measurable sets* $A_1, \ldots, A_n$ *such that*

$$\| \chi_{A_i} f_i \|_F \geq \theta^2 \ .$$

Proof : There is no loss in assuming that $\| \sum_{i=1}^{n} a_i f_i \|_F > \theta \sum_{i=1}^{n} |a_i|$, provided that not all of $a_1, \ldots, a_n$ are zero. Choose now $g \in B_{L_G^*}$, where G is the complement of F, so that

$$| \int g(\sum_{i=1}^{n} a_i f_i) d\mu | > \theta \sum_{i=1}^{n} |a_i| \ .$$

Then

$$\int | \sum_{i=1}^{n} a_i (g f_i) | d\mu > \theta \sum_{i=1}^{n} |a_i|$$

and so by Lemma (3.3.1) there is a collection of measurable and pairwise disjoint sets $A_1, \ldots, A_n$ so that

$$\int_{A_i} |g f_i| d\mu \geq \theta^2 \quad \forall i = 1, \ldots, n \ .$$

By Hölder's inequality we then have that for each $i = 1, \ldots, n$

$$\begin{aligned} \|\chi_{A_i} f_i\|_F &\geq \|g\|_G \cdot \|\chi_{A_i} f_i\|_F \\ &\geq \int_{A_i} |g f_i| d\mu \\ &\geq \theta^2 , \end{aligned}$$

which is what we wanted. ∎

The following theorem, characterizes reflexive subspaces of L_F^*, for $F \in \Delta_2$, with complement G satisfying $\lim_{t\to\infty} \frac{G(mt)}{G(t)} = \infty$

Theorem 3.3.3 *Let $F \in \Delta_2$, with its complement G satisfying*

$$\lim_{t\to\infty} \frac{G(mt)}{G(t)} = \infty$$

for some $m > 0$. Let X be a subspace of L_F^*. Then the following are equivalent :

1. *The space X is not reflexive.*
2. *The space X contains a copy of l_1 complemented in L_F^*.*
3. *The space X contains l_1^n's uniformly.*
4. *The space X fails to have non-trivial type.*

Proof : The implication "1 ⇒ 2" is just theorem (3.1.4). As for "2 ⇒ 3" it follows directly from the definitions. The double implication "3 ⇔ 4" is Pisier's theorem. So we will only show "3 ⇒ 1".

Suppose that X contains l_1^n's uniformly. Then there is a $0 < \theta < 1$ so that for each positive integer n, there are functions $f_1, \ldots, f_n \in B_X$ satisfying

$$\| \sum_{i=1}^n a_i f_i \|_F \geq \theta \sum_{i=1}^n |a_i| ,$$

for any choice of scalars $a_1, \ldots, a_n$. So by Lemma (3.3.2), we have that for each positive integer n, there are functions $f_1, \ldots, f_n \in B_X$ and measurable, pairwise disjoint sets $A_1, \ldots, A_n$ so that

$$\|\chi_{A_i} f_i\|_F \geq \theta^2 \quad i = 1, \ldots, n .$$

Since $A_1, \ldots, A_n$ are pairwise disjoint, at least one of them must have μ-measure less than $\frac{1}{n}$. Thus B_X cannot have equi-absolutely continuous norms. Hence by Corollary (2.1.4), B_X is not weakly compact in L_F^* and so X is not reflexive. ∎

Appendix A

THE PRESENCE OF UNIFORM l_1^n's IN $L^1(\mu)$

Lemma 3.3.1 was presented to me in this form by Joe Diestel. So for the purpose of completeness I include its proof here. Our result will be a direct consequence of the following two lemmas.

Lemma A.1 *Suppose that* $f_1, \ldots, f_n \in B_{L^1(\mu)}$ *and for some* $0 < \theta < 1$ *we have*

$$\|\sum_{i=1}^{n} a_i f_i\|_1 \geq \theta \sum_{i=1}^{n} |a_i|,$$

for any choice of scalars $a_1, \ldots, a_n$. *Then*

$$\| \max_{1 \leq i \leq n} |a_i f_i| \, \|_1 \geq \theta^2 \sum_{i=1}^{n} |a_i|,$$

for any $a_1, \ldots, a_n$.

Lemma A.2 *Let* $f_1, \ldots, f_n$ *be non-negative elements of* $B_{L^1(\mu)}$ *and suppose that there is* $c > 0$ *so that*

$$\int_\Omega (\max_{1 \leq i \leq n} a_i f_i) \, d\mu \geq c \sum_{i=1}^{n} a_i,$$

for any non-negative scalars $a_1, \ldots, a_n$. *Then there exist pairwise disjoint measurable sets* $A_1, \ldots, A_n$ *such that*

$$\int_{A_i} f_i \, d\mu \geq c,$$

for $i = 1, \ldots, n$.

Proof of A.1: Let (r_n) denote the sequence of Rademacher functions. Then using Fubini's theorem, it is easy to see that

$$\theta \sum_{i=1}^{n} |a_i| \quad \leq \quad \int_0^1 \|\sum_{i=1}^{n} a_i r_i(t) f_i\|_1 \, dt$$

$$
\begin{aligned}
&= \int_0^1 \int_\Omega |\sum_{i=1}^n a_i r_i(t) f_i(\omega)| \; d\mu(\omega) \; dt \\
&= \int_\Omega \int_0^1 |\sum_{i=1}^n a_i r_i(t) f_i(\omega)| \; dt \; d\mu(\omega) \\
&\le \int_\Omega (\int_0^1 |\sum_{i=1}^n a_i r_i(t) f_i(\omega)|^2 \; dt)^{\frac{1}{2}} \; d\mu(\omega) \\
&\le \int_\Omega (\sum_{i=1}^n |a_i r_i(t) f_i(\omega)|^2)^{\frac{1}{2}} \; d\mu(\omega) \\
&\le \int_\Omega (\max_{1\le i\le n} |a_i f_i(\omega)|)^{\frac{1}{2}} \cdot (\sum_{i=1}^n |a_i f_i(\omega)|)^{\frac{1}{2}} \; d\mu(\omega) \\
&\le (\int_\Omega \max_{1\le i\le n} |a_i f_i(\omega)| \; d\mu(\omega))^{\frac{1}{2}} \cdot (\int_\Omega \sum_{i=1}^n |a_i f_i(\omega)| \; d\mu(\omega))^{\frac{1}{2}} \\
&\le (\sum_{i=1}^n |a_i|)^{\frac{1}{2}} \cdot (\| \max_{1\le i\le n} |a_i f_i| \; \|_1)^{\frac{1}{2}}
\end{aligned}
$$

and so the first lemma is finished. ∎

Proof of A.2: The proof of the second result is more involved and relies on clever usage of the Hahn-Banach and Krein-Milman theorems. We proceed in three parts.

PART I: We first show that there exist non-negative $\varphi_1, \ldots, \varphi_n$ in L^∞ with $\sum_{i=1}^n \varphi_i \le 1$ so that $\int_\Omega \varphi_i f_i \; d\mu \ge c$ for all $1 \le i \le n$.

Let

$$D = \{ \; (\varphi_1, \ldots, \varphi_n) \; \in \; (L^\infty)^n \; : \; \varphi_1, \ldots, \varphi_n \; \ge 0, \; \sum_{i=1}^n \varphi_i \le 1 \; \} \; .$$

View D as a subset of $(L^\infty \oplus \cdots \oplus L^\infty)_{l_\infty^n}$ and note that D is weak*-compact and convex. Define $T \; : \; (L^\infty \oplus \cdots \oplus L^\infty)_{l_\infty^n} \; \to \; l_\infty^n$ by

$$T(\varphi_1, \ldots, \varphi_n) = (\int_\Omega \varphi_1 f_1 \; d\mu, \ldots, \int_\Omega \varphi_n f_n \; d\mu) \; .$$

It is plain that T is weak* to norm continuous and linear. Thus $T(D)$ is a compact convex subset of l_∞^n. Consider now

$$C = \{ \; (c_1, \ldots, c_n) \; \in \; l_\infty^n \; : \; c_i \ge c, \; i = 1, \ldots, n \; \}.$$

Clearly C is a closed and convex subset of l_∞^n. In order to establish Part I we only need to show that $T(D) \cap C \neq \emptyset$.

So suppose that $T(D) \cap C = \emptyset$. Then by the Hahn-Banach theorem there is $\lambda < 1$ and a point $(a_1, \ldots, a_n)$ in the unit sphere of l_1^n such that

$$\sum_{i=1}^n a_i \int_\Omega \varphi_i f_i \, d\mu \leq \lambda < 1 \leq \sum_{i=1}^n a_i c_i$$

for all $(c_1, \ldots, c_n) \in C$, $(\varphi_1, \ldots, \varphi_n) \in D$. Observe that $a_1, \ldots, a_n$ are non-negative and $c \sum_{i=1}^n a_i \geq 1$. Let

$$g = \max_{1 \leq i \leq n} a_i f_i.$$

Choose pairwise disjoint and measurable sets $E_1, \ldots, E_n$ so that on each E_i, $g = a_i f_i$. Then $(\chi_{E_1}, \ldots, \chi_{E_n}) \in D$ and so

$$\lambda < c \sum_{i=1}^n a_i \leq \int_\Omega g \, d\mu = \int_\Omega (\sum_{i=1}^n a_i f_i \chi_{E_i}) d\mu = \sum_{i=1}^n a_i \int_\Omega f_i \chi_{E_i} d\mu \leq \lambda,$$

which is obviously a contradiction.

PART II: Take non-negative $\varphi_1, \ldots, \varphi_n \in L^\infty$ with $\sum_{i=1}^n \varphi_i \leq 1$ and $\int_\Omega \varphi_i f_i \, d\mu \geq c$ for $i = 1, \ldots, n$. We will show that there exist disjointly supported functions $x_1, \ldots, x_n \in L^\infty$, with exactly the same characteristics. That is $x_1, \ldots, x_n$ non-negative with $\sum_{i=1}^n x_i \leq 1$ and $\int_\Omega x_i f_i \, d\mu \geq c$ for $i = 1, \ldots, n$. Look at

$$D_0 = \{ \ (x_1, \ldots, x_n) \ \in D \ : \ \int_\Omega x_i f_i \, d\mu \ = \ \int_\Omega \varphi_i f_i \, d\mu \ \ for \ \ i = 1, \ldots, n\}.$$

D_0 is a non-empty weak*-compact convex subset of D. By the Krein-Milman theorem D_0 has an extreme point say $(x_1, \ldots, x_n)$. We claim that $x_1, \ldots, x_n$ are disjointly supported. For if not then there are $1 \leq i < j \leq n$ and $\eta > 0$ so that $x = x_i \wedge x_j > \eta$ on some set E of positive measure. Now since μ is non-atomic, the span of the set $\{x \cdot \chi_F \ : F \subset E, F \in \Sigma\}$ is infinite dimensional and so it contains a non-zero h with $|h| \leq x$ and $\int_\Omega h f_i \, d\mu = \int_\Omega h f_j \, d\mu = 0$. But now the points $(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_{j-1}, x_j - h, x_{j+1}, \ldots, x_n)$ and

$(x_1, \ldots, x_{i-1}, x_i - h, x_{i+1}, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n)$ are two distinct points of D_0 whose average is $(x_1, \ldots, x_n)$ which is impossible. So Part II is finished.

PART III: Let A_i be the support of x_i. Then for each i we have

$$\begin{aligned} c \leq \int_\Omega \varphi_i f_i \, d\mu &= \int_\Omega x_i f_i \, d\mu \\ &= \int_{A_i} x_i f_i \, d\mu \\ &\leq \int_{A_i} f_i \, d\mu. \ \blacksquare \end{aligned}$$

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